Math 210B Lecture 14 Notes

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1 Symmetric Powers, Exterior Powers, and Determinants

1.1 Symmetric algebras and powers

Let A be a graded R-algebra.

Definition 1.1. A **homogeneous** ideal I of A is an ideal such that $I = \bigoplus_{k=0}^{\infty} \operatorname{gr}^k(I)$, where $\operatorname{gr}^k(I) = I \cap \operatorname{gr}^k(A)$.

Lemma 1.1. *An ideal is homogeneous if and only if it has a set of generators, each of which lies in some* $\operatorname{gr}^k(A)$.

Example 1.1. Let $I = (x^3 - y^2) \subseteq A = R[x, y]$, which is graded by degree. This is not homogeneous, so A/I is not graded.

Let M be an R-module.

Definition 1.2. The **tensor module** is $T(m) = \bigoplus_{k=0}^{\infty} M^{\otimes k}$.

Definition 1.3. The **symmetric algebra** is $S(M) = T(M)/I$, where I is the ideal generated by $m \otimes n - n \otimes m$ for all $m, n \in M$. We call the graded pieces $\operatorname{Symm}^k(M) = \operatorname{gr}^k(S(M))$.

Example 1.2. $S(R^{\oplus n}) = R[x_1, \dots, x_n]$, and $\operatorname{Symm}^k(R^{\oplus n})$ is the set of homogenerous polynomials of degree k in $x_1, \dots, x_n$.

$\operatorname{Symm}^k(M)$ satisfies a universal property.

Proposition 1.1. *For any $\psi : M^k \to L$ which is R-multilinear and symmetric in its variables, there is a unique Ψ such that*

$$\begin{tikzcd} M \times \cdots \times M \arrow[r, "\psi"] \arrow[d] & L \\ \operatorname{Symm}^k(M) \arrow[ur, "\Psi"'] & \end{tikzcd}$$

If $f : M \to N$ is a morphism of R-modules, then $\operatorname{Symm}^k(f) : \operatorname{Symm}^k(M) \to \operatorname{Symm}^k(N)$ sends $m_1 \otimes \cdots \otimes m_k \mapsto \psi(m_1) \otimes \cdots \otimes \psi(m_k)$.

1.2 Exterior algebras and powers

To get antisymmetric instead of symmetric we could try the ideal generated by the $m \otimes n + n \otimes m$. If $n = m$, we get that $2m \otimes m$ is in the ideal, but $m \otimes m$ is not necessarily in the ideal. But we want $\psi(m, m, m, \dots) = 0$. Instead take,

$$J = (\{m \otimes m : m \in M\}).$$

Then

$$J \ni (m+n) \otimes (m+n) - m \otimes m - n \otimes n = m \otimes n + n \otimes m,$$

so we get all the relations we want.

Definition 1.4. The **exterior algebra** on an R-module M is $\bigwedge(M) = T(M)/J = \bigoplus_{k=0}^{\infty} \bigwedge^k(M)$. $\bigwedge^k(M)$ is called the **k-th extenior product** of M.

The k-th exterior product of M is universal for R-bilinear, alternating mpas in k-variables: $\psi(\dots, m, m, \dots) = 0$ for all m. We write the elements as

$$m \wedge \cdots \wedge m_k \in \bigwedge^k(M).$$

Here are some properties:

1. $m_1 \wedge m_2 \wedge m_3 = -m_1 \wedge m_3 \wedge m_2 = m_3 \wedge m_1 \wedge m_2 = \cdots$
2. $\cdots \wedge m \wedge m \wedge \cdots = 0$

A generalization of the first property is the following,

Lemma 1.2. $m_{\sigma(1)} \wedge \cdots \wedge m_{\sigma(k)} = (\mathrm{sign}(\sigma)) m_1 \wedge \cdots \wedge m_k$.

$\bigwedge^k(R^{\oplus n})$ is spanned by $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $2_1, \dots, e_n$ is the standard basis of $R^{\oplus n}$, and $i_1, \dots, i_k \in \{1, \dots, n\}$. In fact, this is spanned by $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $i_1, \dots, i_k$ are distinct, or equivalently, $i_1 < \cdots < i_k$.

Theorem 1.1. *$\bigwedge^k(R^{\oplus n})$ is free on the generators $e_{i_1} \wedge \cdots \wedge e_{i_k}$ with $1 \le i_1 < \cdots < i_k \le n$. In particular,*

$$\dim\left(\bigwedge^k(R^{\oplus n})\right) = \begin{cases} \binom{n}{k} & k \le n \\ 0 & k > n. \end{cases}$$

Proof. Let $M = R^{\oplus n}$. Fix $i_1 < \cdots i_k$. It suffices to show the there exists some $\Phi : \bigwedge^k M \to R$ such that

$$\Psi(e_{i_1} \wedge \cdots \wedge e_{i_k}) = 1, \qquad \Psi(e_{j_1} \wedge \cdots \wedge e_{j_k}) = 0$$

if $j_1 < \cdots < j_k$ and $(i_1, \dots, i_k) \neq (j_1, \dots, j_k)$. We want a map $\psi : M \times \cdots \times M \to R$. Send

$$\psi(e_{j_1}, \dots, e_{j_k}) = \begin{cases} \operatorname{sign}(\sigma) & i_{\sigma(t)} = j_t \,\forall t \\ 0 & \{i_1, \dots, i_k\} \neq \{j_1, \dots, j_k\} \\ 0 & j_1, \dots, j_k \text{ not distinct} \end{cases}$$

If it is alternating on a basis, it is alternating (exercise), so this is well-defined. Then we get a dual basis of the correct size. □

1.3 Determinants

Say M is free with basis $e_1, \dots, e_n$, and $T : M \to M$ is R-linear. This induces $\bigwedge^n(T) : \bigwedge^n(M) \to \bigwedge^n(M)$; this is a map $R \to R$, and it sends $e_1 \wedge \cdots \wedge e_n \mapsto 1$. This is multiplication by some element of R, which we call $\det(T)$. It satisfies $Te_1 \wedge \cdots \wedge Te_n = \det(T) e_1 \wedge \cdots \wedge e_n$.

Definition 1.5. $\det(T)$ is called the **determinant** of T.

Lemma 1.3. *$Tv_1 \wedge \cdots \wedge Tv_n = \det(T) v_1 \wedge \cdots \wedge v_n$.*

Proof. Expand each v_i as a linear combination of the $e_1 \wedge \cdots \wedge e_n$. Then the statement applies to each $Te_1 \wedge \cdots \wedge Te_n$, and we can do the steps in reverse. □

Proposition 1.2. *Let $T, U : M \to M$. Then $\det(T \circ U) = \det(T)\det(U)$.*

Proof.

$$\begin{aligned} \det(TU) e_1 \wedge \cdots \wedge e_n &= TUe_1 \wedge \cdots \wedge TUe_n \\ &= \det(T) Ue_1 \wedge \cdots \wedge Ue_n \\ &= \det(T)\det(U) e_1 \wedge \cdots \wedge e_n. \end{aligned}$$

□

Corollary 1.1. *If $T : M \to M$ is an isomorphism, $\det(T) \in R^\times$.*

Proof. $\det(T)\det(T)^{-1} = 1$ by the proposition. □